5. Linear transformation

transformation

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function

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mapping

$$f(x) = \overset{P_{2nd}}{\overset{\uparrow}{x^2}} \rightarrow \frac{df}{dx} = \overset{P_{1st}}{\overset{\uparrow}{2x}}$$

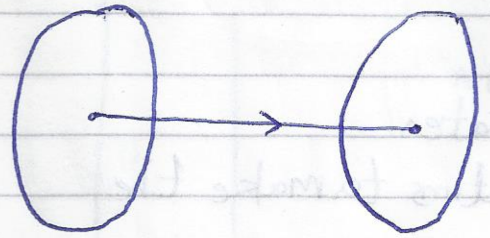
$$\frac{d}{dx}\underset{\text{2nd degree}}{\underset{\downarrow}{f}} \Rightarrow \text{1st degree}$$

$$f(x) = \overset{P_2}{\overset{\uparrow}{x^2}} \Longrightarrow \overset{P_3}{\uparrow}$$

$$\int f(x)\,dx = \int x^2\,dx = \frac{x^3}{3} + c$$

$$\int \underset{\text{2nd degree}}{\underset{\downarrow}{f(x)}} \Longrightarrow \underset{\text{3rd degree}}{\underset{\downarrow}{g(x)}}$$

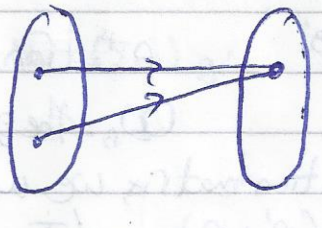
Function



Set(I) Domain

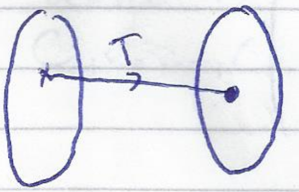
set(II) Codomain/range

or



Domain Codomain

Transformation

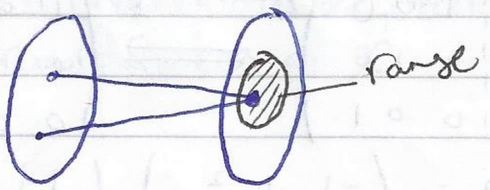


Domain V

Co domain W

or

$\boxed{T: V \rightarrow W}$



Domain Co domain

Range is a subset of the codomain that contains the transformation of all points in the domain.

Ex. $A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 4 \end{pmatrix}$, $V = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$AV = \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = w$$

$$\mathbb{R}^2 \xrightarrow[T]{} \mathbb{R}^3$$

Domain Codomain

$T(V) = w$

Such that $w = AV$

Linear transformation: Let V and W to be vector spaces then the transformation $T: V \to W$ is called a linear transformation of V into W if the following two properties are true:

1. $T(\underline{u}, \underline{v}) = T(\underline{u}) + T(\underline{v})$
2. $T(c\underline{v}) = cT(\underline{v})$, where c = scalar

Ex. $f(x) = \sin x$

$\sin(x_1 + x_2) \neq \sin x_1 + \sin x_2$

$\therefore$ $\sin x$ is not a linear transformation

Ex. $f(x) = x^2$

$f(x_1 + x_2) = (x_1 + x_2)^2 \neq x_1^2 + x_2^2$

$\therefore$ x^2 is not a linear transformation

Ex. $f(x) = x + 1$

$f(x_1) = x_1 + 1$ $f(x_2) = x_2 + 1$ $f(x_1 + x_2) = x_1 + x_2 + 1$

$f(x_1 + x_2) \neq f(x_1) + f(x_2)$

$\therefore$ $x+1$ is not a linear transformation

Ex, Let $T: \mathbb{R}^2 \to \mathbb{R}^2$, such that $T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$

find: ① The image of $v = (-1, 2)$

② The preimage of $w = (-1, 11)$

① $T(v) = T(-1, 2) = (-3, 3)$

② $v_1 - v_2 = -1$

$-\quad v_1 + 2v_2 = 11$

$-3v_2 = -12 \qquad v_2 = 4 \quad v_1 = 3 \qquad$ preimage $(3; 4)$

$T(3,4) = (-1, 11)$

properties of linear transformation

1_ $T(\underline{0}) = \underline{0}$

Ex. let $T: \mathbb{R}^2 \to \mathbb{R}^2$

Such that $T(x_1, x_2) = (x_1 + 1, x_2)$

$T(0,0) \neq (1,0)$

∴ Not a linear transformation

Matrix transformation

Def: Let A be an $m \times n$ matrix, then the transformation T is defined by $T(\underline{v}) = A\underline{v}$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$

Ex $A = \begin{pmatrix} 2 & -3 \\ -5 & 0 \\ 0 & -2 \end{pmatrix} \qquad T: \mathbb{R}^2 \to \mathbb{R}^3$

Ex $B = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 0 & 0 \end{pmatrix} \qquad T: \mathbb{R}^4 \to \mathbb{R}^2$

* Every $m \times n$ matrix represents a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ and Vice versa

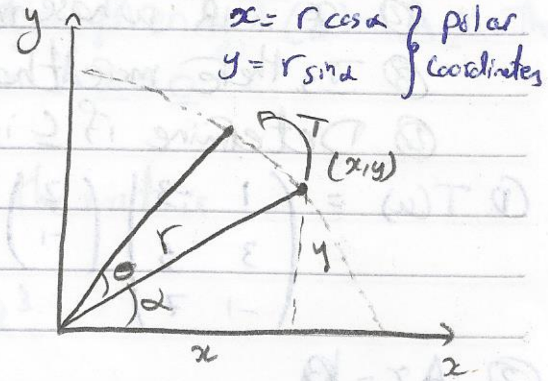
ie. every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be represented by an $m \times n$ matrix

Ex. The matrix $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ represents ~~of~~ a ~~line~~ transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ which has the property that it rotates every vector in $\mathbb{R}^2$ counterclockwise about the origin through an angle θ

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} r\cos\alpha \\ r\sin\alpha \end{pmatrix}$$

$$= \begin{pmatrix} r\cos\theta\cos\alpha - r\sin\theta\sin\alpha \\ r\sin\theta\cos\alpha + r\cos\theta\sin\alpha \end{pmatrix}$$

$$= \begin{pmatrix} r\cos(\theta+\alpha) \\ r\sin(\theta+\alpha) \end{pmatrix}$$

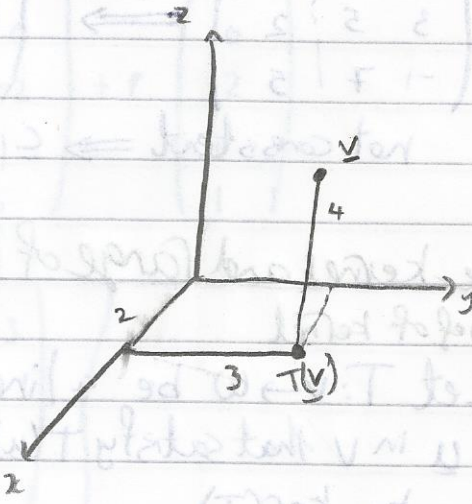


A = rotation matrix

Ex. The matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ represents a transformation

$T: \mathbb{R}^3 \to \mathbb{R}^3$ which is called a projection in $\mathbb{R}^3$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$$



A is a projection matrix

Ex. Let $A = \begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix}$, $\underline{u} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $\underline{b} = \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix}$, $\underline{c} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$

Which defines a transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(\underline{x}) = A\underline{x}$

Find: 1 $T(\underline{u})$

2 $\underline{x}$ in $\mathbb{R}^2$ whose image is $\underline{b}$

3 Is there more than one $\underline{x}$ whose image is $\underline{b}$?

4 Determine if $\underline{c}$ is in the range of T.

1 $T(\underline{u}) = \begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ -9 \end{pmatrix}$

2 $A\underline{x} = \underline{b}$

$(A \vdots b) = \left(\begin{array}{cc|c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{array}\right) \xRightarrow{\text{Echelon}} \left(\begin{array}{cc|c} 1 & 0 & 1.5 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0 \end{array}\right)$ $\quad x_1 = 1.5 \quad x_2 = -0.5$

$\underline{x} = \begin{pmatrix} 1.5 \\ -0.5 \end{pmatrix}$

3 No, because there are no free variables $\Rightarrow$ there is only 1 solution

4 $A\underline{x} = \underline{c}$

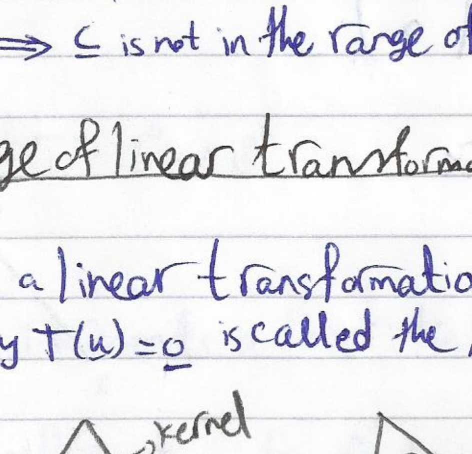
$\left(\begin{array}{cc|c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{array}\right) \xRightarrow{\text{Echelon}} \left(\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -35 \end{array}\right)$

not consistent $\Rightarrow \underline{c}$ is not in the range of T

The kernel and range of linear transformation

Def of kernel

Let $T: V \to W$ be a linear transformation. Then the set of all vectors $\underline{u}$ in V that satisfy $T(\underline{u}) = \underline{0}$ is called the kernel of T and is denoted by $\ker(T)$



Def null space of a matrix

The null space of an $m \times n$ matrix A, written $Nul(A)$, is the set of all solutions of the homogeneous equation $A\underline{x} = \underline{0}$

$$Nul(A) = \{\underline{x} : A\underline{x} = 0\}$$

Theorem: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a transformation given by $T(\underline{x}) = A\underline{x}$. Then the kernel of T is equal to the solution $A\underline{x} = \underline{0}$

Ex Find the spanning set of the nullspace of the matrix

$$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix} \qquad T: \mathbb{R}^5 \to \mathbb{R}^3$$

$$A\underline{x} = \underline{0} \quad \left(\begin{array}{ccccc|c} -3 & 6 & -1 & 1 & -7 & 0 \\ 1 & -2 & 2 & 3 & -1 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{array}\right) \xrightarrow[\text{form}]{\text{echelon}} \left(\begin{array}{ccccc|c} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Let $x_2 = t$, $x_4 = s$, $x_5 = r$

$$x_1 - 2x_2 \quad - x_4 + 3x_5 = 0$$

$$x_3 + 2x_4 - 2x_5 = 0$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2t + s - 3r \\ t \\ -2s + 2r \\ s \\ r \end{pmatrix} = t\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + r\begin{pmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

Spanning set $S = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$